Received: *August 27, 1999*, Accepted: *September 8, 1999*

Holomorphic factorization of correlation functions in $(4k+2)$-dimensional $(2k)$-form gauge theory

Måns Henningson, Bengt E.W. Nilsson and Per Salomonson

Institute of Theoretical Physics
Chalmers University of Technology
S-412 96 Göteborg, Sweden
E-mail: mans@fy.chalmers.se, tfebn@fy.chalmers.se, tfeps@fy.chalmers.se

Abstract: We consider a free $(2k)$-form gauge-field on a euclidean $(4k+2)$-manifold. The parameters needed to specify the action and the gauge-invariant observables take their values in spaces with natural complex structures. We show that the correlation functions can be written as a finite sum of terms, each of which is a product of a holomorphic and an anti-holomorphic factor. The holomorphic factors are naturally interpreted as correlation functions for a chiral $(2k)$-form, i.e. a $(2k)$-form with a self-dual $(2k+1)$-form field strength, after Wick rotation to a Minkowski signature.

Keywords: Field Theories in Higher Dimensions, Differential and Algebraic Geometry.

JHEP09(1999)008

Contents

1. Introduction

Higher rank abelian gauge fields play a prominent role in string theory, although their geometrical meaning is not quite clear. Locally, such a field is given by a p-form B modulo the gauge equivalence relation $B \sim B + \Delta B$, where ΔB is a closed p-form with integer periods. When $p = 1$, this is just an ordinary abelian gauge field with its usual gauge invariance.

On a d-manifold M, a natural class of gauge invariant observables can be written in the form $\exp 2\pi i \int_M B \wedge P$, where P is a closed $(d-p)$-form with integer periods. For example, if P is the Poincaré dual of some p-cycle Σ in M, we get the 'Wilson volume' observable $\exp 2\pi i \int_\Sigma B$ associated with Σ. A product of such observables is again of the same form with the resulting parameter P being the sum of the P's of the different factors. To compute the expectation value of such an observable, it is in fact sufficient to consider the case when P is trivial in cohomology, i.e. when $P = 2dQ$ for some $(d-p-1)$-form Q. (The factor of 2 is inserted for later convenience.) For example, P could be twice the Poincaré dual of a homologically trivial p-cycle Σ in M. The observable is then of the form

$$W(Q) = \exp 4\pi i \int_M H \wedge Q\,, \tag{1.1}$$

where $H = dB$ is the gauge invariant $(p+1)$-form field strength of B. The reason that we do not need to consider cohomologically non-trivial P is that in this case, the B-field would contain a closed mode Poincaré dual to P. Furthermore, this mode does not appear in the action that we will discuss shortly. Because of the gauge invariance, the coefficient of this mode takes its values on a circle. The expectation value of the observable then vanishes, because its phase averages to zero when we integrate over this mode. This phenomenon is well known for a compact scalar in two dimensions, where it corresponds to conservation of discrete momentum (in a string theory compactification).

If M is endowed with a metric G, it is natural to consider the generalized Maxwell action

$$S = -\frac{2\pi}{g^2}\int_M H \wedge {}^*H\,, \tag{1.2}$$

where * denotes the Hodge duality operator that maps a $(p+1)$-form to a $(d-p-1)$-form, and g is a coupling constant. The classical equation of motion that follows from this action is

$$d^*H = 0\,, \tag{1.3}$$

i.e. a field strength H that fulfills the classical equation of motion is harmonic. We can now calculate the (unnormalized) expectation value of the observable $W(Q)$ as

$$\langle W(Q)\rangle = \int \mathcal{D}B\, W(Q) e^{-S}\,, \tag{1.4}$$

where the functional integral is over gauge inequivalent field configurations B.

In this paper, we will be concerned with the case when $d = 4k+2$ and $p = 2k$ for some integer k. A priori, we are interested in the case when the metric G has a Minkowski signature, which means that the Hodge duality operator * obeys $^{**} = 1$, so that its eigenvalues are ± 1. We now say that a gauge field B is chiral if its field strength $H = dB$ obeys the self-duality equation

$$H = +{}^*H\,. \tag{1.5}$$

From this first order differential equation in fact follows the second-order equation of motion (1.3). Similarly, an anti-chiral gauge field has an anti self-dual field strength. Chiral gauge fields have many important applications in string theory. For example, in $d = 2$ dimensions, chiral zero-form (scalars) appear on the world-sheet of heterotic string theory. In $d = 6$ dimensions, the world-volume theory of the M-theory five-brane contains a chiral two-form. Finally, in $d = 10$ dimensions, type IIB supergravity contains a chiral four-form. In $d = 2$ dimensions, there is a well-known interacting generalization of the theory, namely the chiral Wess-Zumino-Witten model. In $d = 6$ dimensions, the tensionless string theories generalize the theory of a free $(0,2)$ tensor multiplet (which contains a chiral two-form). In $d = 10$ dimensions, no interacting generalization is known.

JHEP09(1999)008

One can show that the self-duality equation (1.5) cannot follow from any covariant lagrangean in the usual sense. It is therefore a bit subtle to define the quantum theory of a chiral gauge field [2]. The main idea is as follows: One starts with the non-chiral theory with action (1.2). The field strength H then contains both a self-dual and an anti self-dual part. However, since the theory is free, i.e. the action is a bilinear in the gauge field B, one would expect that the chiral and anti-chiral parts are decoupled from each other. Furthermore, observables of the form (1.1) are exponentials of linear expressions in B. One would therefore expect that the correlation function (1.4) can be written as the product of the correlation functions pertaining to a chiral gauge field and an anti-chiral gauge field. (We will explain the criterion for how to perform this factorization in the next section.) On a manifold M of non-vanishing middle dimensional Betti number $b_{2k+1} = \dim H^{2k+1}(M, \mathbb{R})$, the story is in fact more complicated. It will turn out that there are then $2^{b_{2k+1}}$ different candidate chiral and anti-chiral correlation functions. Assembling these into vectors $\langle W(Q) \rangle_+$ and $\langle W(Q) \rangle_-$ respectively, one can write the non-chiral correlation function as

$$\langle W(Q) \rangle = \langle W(Q) \rangle_+ \cdot \langle W(Q) \rangle_- \ , \tag{1.6}$$

where the raised dot denotes the vector scalar product. To define the correlation function of a chiral gauge-field, one would thus need some extra discrete data to pick out one of the terms in (1.6). In applications, this choice is in fact often dictated by the physical context [2]. (On a torus, it turns out that there is a canonical choice which is invariant under the $\mathrm{SL}(4k+2, \mathbb{Z})$ mapping class group. The corresponding chiral partition function was calculated in the case of a flat metric in [3].)

In this paper, we will carry out the procedure outlined in [2] in detail and derive (1.6) explicitly. We will consider the exact quantum correlation function for a general observable of the form (1.1). (The discussion in [2] focuses on the partition function and the contributions to the functional integral (1.4) from field configurations that solve the classical equations of motion (1.3). In that paper, the theory is coupled to a background $(2k+1)$-form, that is put to zero in this paper.) In the next section, we will describe how to separate a non-chiral correlation function into its chiral and anti-chiral parts. In fact, the non-chiral correlation function also contains certain 'anomalous' factors that have to be discarded in order for the factorization (1.6) to work. In section 3, we will evaluate the contributions due to classical field configurations, and in the last section we will consider the quantum fluctuations.

2. Holomorphic factorization

2.1 The complex structure

We thus consider the case of a $(2k+1)$-form gauge field B in a $(4k+2)$-dimensional space M. The coupling constant g in (1.2) is then dimensionless, so that the action is scale invariant and in fact only depends on the conformal equivalence class $[G]$ of the metric G on M. (Our treatment in this paper is purely formal, and we will thus neglect conformal anomalies such as the one discussed in [1].) Actually, $[G]$ only appears in (1.2) through the Hodge duality operator * that maps the space

$$\Omega = \{(2k+1) - \text{forms on } M\}\,, \tag{2.1}$$

of which the field strength H is an element, to itself. We note that the parameter Q in the observable (1.1) is also an element of Ω.

To describe how to separate the non-chiral correlation function (1.4) into its chiral and anti-chiral parts, we will actually have to assume that the metric G (and thus the conformal structure $[G]$) of M has a euclidean signature. The Hodge duality operator * then obeys $^{**} = -1$, and can thus be thought of as defining a complex structure J on the space Ω defined in (2.1). The moduli space $\mathcal{J}$ of all complex structures on Ω itself carries a natural complex structure. It is easy to show that the map described above from the moduli space $\mathcal{G}$ of conformal structures on M to $\mathcal{J}$ is injective, but not surjective. As far as we know, the image of this map, which can thus be identified with $\mathcal{G}$, does not carry a canonical complex structure.

Here it is in place to make some comments on the much-studied case $d = 2$, which has some additional simplifying features. In this dimension, the space Ω is simply the cotangent space, and it is well-known that a choice of a euclidean conformal structure $[G]$ on M induces a complex structure not only on Ω but on M itself. Furthermore, the moduli space $\mathcal{J}$ of such complex structures is finite-dimensional. However, these special properties are not necessary for our discussion, and in the sequel we will consider the general case of $d = 4k+2$ for arbitrary k.

To describe the complex structure J on the space Ω of $(2k+1)$-forms on M more concretely, we begin by noting that Ω carries a natural symplectic structure. The symplectic product is simply given by the wedge product followed by integration over M. We can choose a basis $(E_A)_i$ and $(E_B)_i$ of Ω that is symplectic in the sense that

$$\begin{aligned}\int_M (E_A)_i \wedge (E_A)_j &= \int_M (E_B)_i \wedge (E_B)_j = 0 \\ \int_M (E_B)_i \wedge (E_A)_j &= -\int_M (E_A)_i \wedge (E_B)_j = \delta_{ij}\,.\end{aligned} \tag{2.2}$$

Henceforth we will use a more compact notation where the basis elements $(E_A)_i$ and $(E_B)_j$ are assembled into two (infinite dimensional) column vectors E_A and E_B. The above conditions can then be written as

$$\begin{aligned}\int_M E_A \wedge E_A^t &= \int_M E_B \wedge E_B^t = 0 \\ \int_M E_B \wedge E_A^t &= -\int_M E_A \wedge E_B^t = 1\,,\end{aligned} \tag{2.3}$$

where the 1 on the right hand side of the second equation denotes the unit matrix and ^{t} denotes the transpose. An analogous notation will be employed for other spaces throughout this paper. An element of Ω, such as the field strength H, can be expanded in this basis as

$$H = H_A^t E_A + H_B^t E_B \,, \tag{2.4}$$

with some coefficient vectors H_A and H_B. The same of course applies to the $(2k+1)$-form Q that enters in the observable (1.1).

Sofar, everything has been independent of the Hodge duality operator *. For a generic choice of *, the entries of any two of the vectors E_A, E_B, $^*E_A$, and $^*E_B$ span Ω. We can for example expand E_B as a linear combination of E_A and $^*E_A$, i.e.

$$E_B = X E_A + Y {}^*E_A \,. \tag{2.5}$$

One can show that the real coefficient matrices X and Y are symmetric, and with a suitable choice of E_A and E_B (subject to the relations above), Y is positive definite. The Hodge duality operator *, and thus the complex structure J on Ω, is completely determined by X and Y, which thus can be regarded as coordinates on the moduli space $\mathcal{J}$ of complex structures on Ω. The complex structure on $\mathcal{J}$ can be described by declaring that the complex linear combination

$$Z = X + iY \tag{2.6}$$

is a holomorphic coordinate on $\mathcal{J}$. According to the above, it fulfills the conditions

$$\begin{aligned} Z &= Z^t \\ \mathrm{Im}\, Z &> 0 \,. \end{aligned} \tag{2.7}$$

It is often convenient to change basis for Ω from E_A and E_B to a holomorphic and anti-holomorphic basis E_+ and E_- defined as

$$\begin{aligned} E_+ &= (Z - \bar{Z})^{-1}(E_B - \bar{Z} E_A) \\ E_- &= -(Z - \bar{Z})^{-1}(E_B - Z E_A) \,. \end{aligned} \tag{2.8}$$

which fulfill $^*E_+ = +iE_+$ and $^*E_- = -iE_-$. We can then expand for example the field strength H (or the parameter Q) as

$$H = H_+^t E_+ + H_-^t E_- \,, \tag{2.9}$$

where the coefficient vectors H_+ and H_- are related to H_A and H_B as

$$\begin{aligned} H_+ &= H_A + Z H_B \\ H_- &= H_A + \bar{Z} H_B \,. \end{aligned} \tag{2.10}$$

2.2 Holomorphicity of $\langle W(Q)\rangle_+$

We will now describe a criterion for distinguishing between the contributions to the correlation functions from a chiral and an anti-chiral gauge field.

With the notation introduce above, we can write the action (1.2) as

$$S = \frac{4\pi i}{g^2}(H_A + ZH_B)^t(Z-\bar{Z})^{-1}(H_A+\bar{Z}H_B)\,. \tag{2.11}$$

Remembering that H_A and H_B are independent of $\bar{Z}$, we find that under an infinitesimal anti-holomorphic variation $\delta\bar{Z}$,

$$\delta S = \frac{4\pi i}{g^2}(H_A + ZH_B)^t(Z-\bar{Z})^{-1}\delta\bar{Z}(Z-\bar{Z})^{-1}(H_A+ZH_B)\,. \tag{2.12}$$

Since the observable (1.1) is independent of $\bar{Z}$ we then get, by differentiating under the integration sign in (1.4) that $\delta\langle W(Q)\rangle = -\langle W(Q)\delta S\rangle$. For a chiral gauge field, $H_+ = H_A + ZH_B$ is zero, and the above expression for δS would thus vanish identically. It is then natural to conjecture that the correlation function for such a theory should be holomorphic, i.e.

$$\frac{\delta}{\delta\bar{Z}}\langle W(Q)\rangle_+ = 0\,. \tag{2.13}$$

Similarly, the correlation function $\langle W(Q)\rangle_-$ of an anti-chiral gauge field should be anti-holomorphic. The requirement that the non-chiral correlation function $\langle W(Q)\rangle$ in (1.6) can be holomorphically factorized in this manner is a non-trivial constraint, that we will verify in the remainder of this paper. (In fact, we will see that this is only true if certain non-holomorphic 'anomalous' factors are discarded.) In $d=2$ dimensions, such a holomorphic factorization is an important and well-known feature of many conformal field theories, e.g. Wess-Zumino-Witten models and cosets thereof [4].

2.3 Classical and quantum contributions

We can decompose the space Ω of $(2k+1)$-forms as

$$\Omega = \Omega^0 \oplus \Omega'\,, \tag{2.14}$$

where Ω^0 is the (finite-dimensional) subspace of harmonic forms, and Ω' its orthogonal complement (with respect to the symplectic metric). The Hodge duality operator * respects this decomposition. The field strength H and the parameter Q of the observable (1.1) can accordingly be decomposed as

$$\begin{aligned} H &= H^0 + H' \\ Q &= Q^0 + Q'\,. \end{aligned} \tag{2.15}$$

By the Hodge-de Rham theorem, H^0 is then the unique harmonic representative of the cohomology class $[H]$ of H, whereas $H' = dB'$ for some globally defined $2k$-form B'. In view of (1.3), it is natural to regard H' as a quantum fluctuation around a classical field configuration H^0. Inserting (2.15) into the formulas (1.2) and (1.1), we then get $S = S^0 + S'$ and $W(Q) = W(Q^0)W(Q')$, where

$$\begin{aligned} S^0 &= -\frac{2\pi}{g^2}\int_M H^0 \wedge {}^*H^0 \\ S' &= -\frac{2\pi}{g^2}\int_M H' \wedge {}^*H' \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} W(Q^0) &= \exp 4\pi i \int_M H^0 \wedge Q^0 \\ W(Q') &= \exp 4\pi i \int_M H' \wedge Q' \,. \end{aligned} \tag{2.17}$$

In this way, we get $\langle W(Q)\rangle = \langle W(Q^0)\rangle \langle W(Q')\rangle$, where the classical and quantum contributions are given by

$$\big\langle W(Q^0)\big\rangle = \sum_{H^0} W(Q^0) e^{-S^0} \tag{2.18}$$

and

$$\langle W(Q')\rangle = \int \mathcal{D}B' W(Q') e^{-S'} \tag{2.19}$$

respectively. The sum in the classical part is over harmonic $(2k+1)$-forms H^0 with integer periods, and the functional integral in the quantum part is over globally defined $(2k)$-forms B' modulo closed forms with integer periods.

3. The classical contribution

3.1 The Poisson resummation

The classical field strength H^0 belongs to the lattice $\Gamma \subset \Omega^0$ of harmonic $(2k+1)$-forms with integer periods. We now introduce a basis E^0_A and E^0_B for this lattice, which is symplectic in the same sense as E_A and E_B in (2.2). A difference is of course that E^0_A and E^0_B are finite dimensional vectors with $\frac{1}{2}b_{2k+1} = \frac{1}{2}\mathrm{dim H^{2k+1}(M,\mathbb{R})}$ entries each. By a reasoning exactly analogous to that in the paragraph following (2.4), we see that the Hodge duality operator * induces a conformal structure J^0 parametrized by $Z^0 = X^0 + iY^0$ on the space Ω^0 of harmonic $(2k+1)$-forms. We can then also introduce the corresponding holomorphic and anti-holomorphic basis E^0_+ and E^0_- of Ω^0.

The classical field strength H^0 can now be expanded as

$$\begin{aligned} H^0 &= H^{0t}_A E^0_A + H^{0t}_B E^0_B \\ &= H^{0t}_+ E^0_+ + H^{0t}_- E^0_- \,, \end{aligned} \tag{3.1}$$

where the coefficient vectors H_A^0 and H_B^0 have integer entries. Similarly, we expand the harmonic part Q^0 of the observable parameter as

$$\begin{aligned} Q^0 &= Q_A^{0t} E_A^0 + Q_B^{0t} E_B^0 \\ &= Q_+^{0t} E_+^0 + Q_-^{0t} E_-^0 \,. \end{aligned} \tag{3.2}$$

Inserting these expansions in (2.16)–(2.18) and fixing the coupling constant g at the particular value $g = 1$, we get

$$\begin{aligned} \langle W(Q^0)\rangle = \sum_{H_A^0, H_B^0} \exp\Big[-2\pi\Big(& H_A^{0t}(Y^0)^{-1}H_A^0 + 2H_A^{0t}(Y^0)^{-1}X^0 H_B^0 + \\ &+ H_B^{0t}(X^0(Y^0)^{-1}X^0 + Y^0)H_B^0\Big) + \\ &+ 4\pi i\left(H_B^{0t}Q_A^0 - H_A^{0t}Q_B^0\right)\Big] \,. \end{aligned} \tag{3.3}$$

We now perform a Poisson resummation, replacing the sum over H_A^0 with a sum over a vector m with integer entries. Renaming the vector H_B^0 as n, we then get

$$\begin{aligned} \langle W(Q^0)\rangle = \sqrt{\det \frac{1}{2}Y^0}\sum_{n,m} \exp -2\pi\Big(& n^t Y^0 n - i m^t X^0 n + \frac{1}{4} m^t Y^0 m - 2 i n^t Q_A^0 - \\ &- 2 i n^t X^0 Q_B^0 + m^t Y^0 Q_B^0 + Q_B^{0t} Y^0 Q_B^0\Big) \,, \end{aligned} \tag{3.4}$$

or, in terms of complex variables,

$$\begin{aligned} \langle W(Q^0)\rangle = \sqrt{\det \frac{1}{4i}(Z^0 - \bar{Z}^0)}\sum_{n,m} \exp i\pi\Big(&\left(\frac{m}{2}+n\right)^t Z^0 \left(\frac{m}{2}+n\right) - \\ - \left(\frac{m}{2}-n\right)^t \bar{Z}^0 \left(\frac{m}{2}-n\right) &+ 2\left(\frac{m}{2}+n\right)^t Q_+^0 - 2\left(\frac{m}{2}-n\right)^t Q_-^0 + \\ +(Q_-^0 - Q_+^0)^t (Z^0 - \bar{Z}^0)^{-1}&(Q_-^0 - Q_+^0)\Big) \,. \end{aligned} \tag{3.5}$$

3.2 A digression on theta-functions

The coefficients Q_A^0 and Q_B^0 can be regarded as real coordinates on the space Ω^0 of harmonic $(2k+1)$-forms. We will be interested in the torus Ω^0/Γ (the intermediate jacobian of the manifold M), where Γ is the lattice of forms with integer periods. This means that we should identify

$$\begin{aligned} Q_A^0 &\sim Q_A^0 + \lambda_A^0 \\ Q_B^0 &\sim Q_B^0 + \lambda_B^0 \end{aligned} \tag{3.6}$$

for any vectors λ_A^0 and λ_B^0 with integer entries. The natural complex structure on Ω^0 means that Ω^0/Γ can be regarded as the complex torus $\mathbf{C}^{\frac{1}{2}b_{2k+1}}/\Gamma$ with a holomorphic

JHEP09(1999)008

coordinate Q^0_+ subject to the identifications

$$\begin{aligned} Q^0_+ &\sim Q^0_+ + \lambda^0_A \\ Q^0_+ &\sim Q^0_+ + Z^0\lambda^0_B\,. \end{aligned} \tag{3.7}$$

The parameter Z^0 obeys conditions analogous to (2.7), which implies that Ω^0/Γ is an abelian variety [5]. It is endowed with a (principal) polarization given by the cohomology class $[\omega]$ of the symplectic form ω corresponding to the natural symplectic structure on Ω^0.

Consider now a unitary line bundle $\mathcal{L}$ over Ω^0/Γ with a connection whose curvature equals ω. Since ω is of type $(1,1)$, $\mathcal{L}$ is in fact a holomorphic line bundle. As a complex line bundle, $\mathcal{L}$ is determined by its Chern class $c_1(\mathcal{L}) = [\omega]$, but there is a finer classification of holomorphic line bundles [5]. Indeed, the kernel of the Chern class c_1 is isomorphic to the complex torus $H^1(\Omega^0/\Gamma, \mathcal{O})/H^1(\Omega^0/\Gamma, \mathbb{Z})$ (the Picard variety or the dual abelian variety), where $\mathcal{O}$ is the sheaf of holomorphic functions. The set of holomorphic line bundles $\mathcal{L}$ with $c_1(\mathcal{L}) = [\omega]$ is thus also a torus, which we can parametrize by α and β that are vectors of dimension $\frac{1}{2}b_{2k+1}$ with entries with values in $\mathbb{R}/\mathbb{Z}$. We denote the corresponding line bundle as $\mathcal{L}[^\alpha_\beta]$.

It follows from an index theorem together with the Kodaira vanishing theorem that $\mathcal{L}[^\alpha_\beta]$ has a unique holomorphic section (up to a multiplicative constant). In a certain trivialization of $\mathcal{L}[^\alpha_\beta]$, this is given by the Jacobi theta-function

$$\theta\left[^\alpha_\beta\right](Z^0|Q^0_+) = \sum_k \exp i\pi\left((k+\alpha)^t Z^0(k+\alpha) + 2(k+\alpha)^t(Q^0_+ + \beta)\right)\,, \tag{3.8}$$

where the sum runs over vectors k of dimension $\frac{1}{2}b_{2k+1}$ with integer entries. The theta-function obeys the quasi-periodicity conditions

$$\begin{aligned} \frac{\theta\left[^\alpha_\beta\right](Z^0|Q^0_+ + \lambda^0_A)}{\theta\left[^\alpha_\beta\right](Z^0|Q^0_+)} &= \exp 2\pi i\alpha^t\lambda^0_A \\ \frac{\theta\left[^\alpha_\beta\right](Z^0|Q^0_+ + Z^0\lambda^0_B)}{\theta\left[^\alpha_\beta\right](Z^0|Q^0_+)} &= \exp i\pi\left(-\lambda^0_B Z^0\lambda^0_B - 2\lambda^{0t}_B(Q^0_+ + \beta)\right)\,, \end{aligned} \tag{3.9}$$

so in this trivialization, the transition functions are holomorphic $\mathbf{C}^*$-valued functions. We will use a different trivialization, though, where the unique holomorphic section of $\mathcal{L}[^\alpha_\beta]$ is given by the function

$$\Theta\left[^\alpha_\beta\right](Z^0|Q^0_+, Q^0_-) = \exp i\pi Q^{0t}_+ Q^0_B\,\theta\left[^\alpha_\beta\right](Z^0|Q^0_+)\,. \tag{3.10}$$

Its quasi-periodicity properties are

$$\begin{aligned} \frac{\Theta\left[^\alpha_\beta\right](Z^0|Q^0_+ + \lambda^0_A, Q^0_- + \lambda^0_A)}{\Theta\left[^\alpha_\beta\right](Z^0|Q^0_+, Q^0_-)} &= \exp i\pi\lambda^{0t}_A(Q^0_B + 2\alpha) \\ \frac{\Theta\left[^\alpha_\beta\right](Z^0|Q^0_+ + Z^0\lambda^0_B, Q^0_- + \bar{Z}^0\lambda^0_B)}{\Theta\left[^\alpha_\beta\right](Z^0|Q^0_+, Q^0_-)} &= \exp -i\pi\lambda^{0t}_B(Q^0_A + 2\beta)\,, \end{aligned} \tag{3.11}$$

so in this trivialization, the transition functions are $U(1)$-valued. Finally we note that $\Theta[{\alpha \atop \beta}](Z^0|Q^0_+, Q^0_-)$ is holomorphic in the sense that it is annihilated by $\delta/\delta \bar{Z}^0$ and by the covariant derivative $D/DQ^0_- = \delta/\delta Q^0_- + i\pi Q^{0t}_+(Z^0 - \bar{Z}^0)^{-1}$.

3.3 Holomorphic factorization

We can now rewrite the expression (3.5) for $\langle W(Q^0)\rangle$ in terms of the functions $\Theta[{\alpha \atop \beta}](Z^0|Q^0_+, Q^0_-)$ introduced in (3.10):

$$\left\langle W(Q^0)\right\rangle = 2^{-\frac{1}{2}b_{2k+1}} \sqrt{\det \frac{1}{4i}(Z^0 - \bar{Z}^0)} \sum_{\alpha\beta} \Theta \left[{\alpha \atop \beta}\right] (Z^0|Q^0_+, Q^0_-) \overline{\Theta \left[{\alpha \atop \beta}\right] (Z^0|Q^0_+, Q^0_-)}\,, \tag{3.12}$$

where the sum over α and β runs over vectors with entries 0 or 1/2. This formula can be verified as follows: We first perform the sum over β, which gives rise to the factor $2^{\frac{1}{2}b_{2k+1}}$ and restricts the sums over k in the definition (3.8) of $\theta[{\alpha \atop \beta}](Z^0|Q^0_+)$ and the corresponding vector l in the complex conjugate to values such that $k - l$ is a vector with even entries. The sums over k and l can then be exchanged to sums over the vectors m' and n with integer entries defined through the relations $k = m' + n$ and $l = m' - n$. Finally, the sums over m' and α can be exchanged to a sum over the vector $m = 2(m' + \alpha)$, which yields (3.5).

We must now pick one of the terms in (3.12), i.e. a particular value of α and β. The chiral part of the classical correlation function is then

$$\left\langle W(Q^0)\right\rangle_+ = \Theta \left[{\alpha \atop \beta}\right] (Z^0|Q^0_+, Q^0_-)\,, \tag{3.13}$$

and the anti-chiral part is its complex conjugate. This leaves the 'anomalous' prefactor

$$\left\langle W(Q^0)\right\rangle_{anom} = 2^{-\frac{1}{2}b_{2k+1}} \sqrt{\det \frac{1}{4i}(Z^0 - \bar{Z}^0)} \tag{3.14}$$

in (3.12), which is neither holomorphic nor anti-holomorphic.

4. The quantum contribution

The quantum part of the gauge field is given by a globally defined $(2k)$-form B' modulo closed forms. Introducing a basis F in the space of such forms, we can then expand B' as

$$B' = b^t F \tag{4.1}$$

for some vector b of coefficients. The exterior derivative d acting on F gives a vector dF of exact $(2k+1)$-forms, which can be expressed in terms of the symplectic basis E_A and E_B or the holomorphic basis E_+ and E_-:

$$dF = \Pi^t_A E_A + \Pi^t_B E_B = \Pi^t_+ E_+ + \Pi^t_- E_- \tag{4.2}$$

for some matrices Π_A, Π_B, Π_+ and Π_- of coefficients. The quantum part $H' = dB'$ of the field strength is then

$$H' = b^t \left(\Pi_A^t E_A + \Pi_B^t E_B\right), \tag{4.3}$$

and the quantum part of the action (2.16) is

$$S' = \frac{4\pi i}{g^2} b^t K b\,, \tag{4.4}$$

where the symmetric matrix K is given by

$$K = \Pi_+^t \left(Z - \bar{Z}\right)^{-1} \Pi_-\,. \tag{4.5}$$

Expanding the quantum part Q' of Q in (2.15) as

$$Q' = Q_+^t E_+ + Q_-^t E_- \tag{4.6}$$

we get for the quantum part of the observable (2.17)

$$W(Q') = \exp 4\pi i b^t J\,, \tag{4.7}$$

where

$$J = \left(\Pi_+^t (Z - \bar{Z})^{-1} Q_- - \Pi_-^t (Z - \bar{Z})^{-1} Q_+\right). \tag{4.8}$$

We can now perform the gaussian path-integral over the B'-field in (2.19), i.e. over the coefficients b, with the result

$$\langle W(Q')\rangle \sim (\det K)^{-1/2} \exp g^2 \pi i J^t K^{-1} J\,. \tag{4.9}$$

For this formula to make sense, the matrix K in (4.5) must be invertible. If we furthermore assume that the matrices Π_+ and Π_+ are also invertible, we can simplify further to get

$$\begin{aligned}\langle W(Q')\rangle \sim \left(\det \Pi_+^t (Z - \bar{Z})^{-1} \Pi_-\right)^{-1/2} \exp i\pi \Big(& Q_+^t (Z - \bar{Z})^{-1} \Pi_- \Pi_+^{-1} Q_+ - \\ & -2 Q_-^t (Z - \bar{Z})^{-1} Q_+ + Q_-^t (Z - \bar{Z})^{-1} \Pi_+ \Pi_-^{-1} Q_- \Big)\,,\end{aligned} \tag{4.10}$$

where we have also put $g = 1$. Finally, we write the result in factorized form as

$$\langle W(Q')\rangle = \langle W(Q')\rangle_{anom} \langle W(Q')\rangle_+ \langle W(Q')\rangle_-\,, \tag{4.11}$$

where the chiral part is given by the holomorphic expression

$$\langle W(Q')\rangle_+ = (\det \Pi_+)^{-1/2} \exp(-i\pi) Q_+^t \Pi_B \Pi_+^{-1} Q_+ \tag{4.12}$$

and the anti-chiral part by its complex conjugate whereas the 'anomalous' prefactor is

$$\langle W(Q')\rangle_{anom} = \left(\det(Z - \bar{Z})\right)^{1/2} \exp i\pi (Q_+^t - Q_-^t)(Z - \bar{Z})^{-1} (Q_+ - Q_-)\,. \tag{4.13}$$

Putting everything together, the final answer for the chiral correlation function is thus

$$\langle W(Q)\rangle_+ = (\det \Pi_+)^{-1/2} \exp(-i\pi) Q_+^t \Pi_B \Pi_+^{-1} Q_+ \, \Theta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (Z^0|Q_+^0, Q_-^0)\,, \qquad (4.14)$$

where $\Theta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (Z^0|Q_+^0, Q_-^0)$ is defined by (3.10) and (3.8).

It remains to discuss the non-holomorphic factors (3.14) and (4.13). These must be discarded in order for the non-chiral correlation function to admit a holomorphic factorization. To make sense of the infinite-dimensional determinant in (4.13), we can employ for example ζ-function regularization. On a flat $(4k+2)$-torus (generalizing the two-dimensional calculation in [6]), it turns out this determinant cancels the finite-dimensional determinant in (3.14). It would be interesting to know if this is true also in more general situations. The remaining factor in (4.13) is in some sense a 'contact term' since it does not involve the 'derivative' matrices Π_+ and Π_-.

This research was supported by the Swedish Natural Science Research Council (NFR).

References

[1] M. Henningson and K. Skenderis, *Weyl anomaly for Wilson surfaces*, *J. High Energy Phys.* **06** (1999) 012 [hep-th/9905163].

[2] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [hep-th/9610234].

[3] L. Dolan and C. Nappi, *A modular invariant partition function for the five-brane*, *Nucl. Phys.* **B 530** (1998) 683 [hep-th/9806016].

[4] E. Witten, *On holomorphic factorization of WZW and coset models*, *Comm. Math. Phys.* **144** (1992) 189.

[5] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley 1978.

[6] J. Polchinski, *Evaluation of the one-loop string path integral*, *Comm. Math. Phys.* **104** (1986) 37.